Problem 2.10

A charge q sits at the back corner of a cube, as shown in Fig. 2.17. What is the flux of $\mathbf{E}$ through the shaded side?

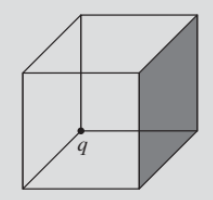
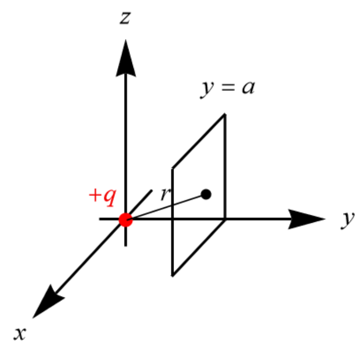


Fig. 2.17

Solution

Choose the coordinate system so that the charge is at the origin and the shaded plane lies perpendicular to the y-axis. Let the cube side length be a.



The electric field of a point charge is known, so the electric flux through the shaded side $\mathcal{S}$ can be calculated.

$$\begin{aligned}
\Phi_E &= \iint \mathbf{E} \cdot d\mathbf{S} \\
&= \iint_{\mathcal{S}} \left(\frac{1}{4\pi\epsilon_0} \frac{q}{r^2} \hat{\mathbf{r}} \right) \cdot (\hat{\mathbf{y}}\, dS) \\
&= \int_0^a \int_0^a \left(\frac{1}{4\pi\epsilon_0} \frac{q}{x^2+y^2+z^2} \frac{x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}}{\sqrt{x^2+y^2+z^2}} \right) \cdot (\hat{\mathbf{y}}\, dx\, dz) \\
&= \int_0^a \int_0^a \frac{1}{4\pi\epsilon_0} \frac{q}{x^2+y^2+z^2} \frac{y}{\sqrt{x^2+y^2+z^2}}\, dx\, dz \\
&= \frac{q}{4\pi\epsilon_0} \int_0^a \int_0^a \frac{y}{(x^2+y^2+z^2)^{3/2}}\, dx\, dz \\
&= \frac{q}{4\pi\epsilon_0} \int_0^a \int_0^a \frac{a}{(x^2+a^2+z^2)^{3/2}}\, dx\, dz \qquad \text{(On the surface } \mathcal{S},\ y = a.)
\end{aligned}$$

Write the double integral as an iterated integral.

$$\Phi_E = \frac{qa}{4\pi\epsilon_0}\int_0^a \left[\int_0^a \frac{dx}{(x^2+a^2+z^2)^{3/2}}\right] dz$$

Make the following substitution.

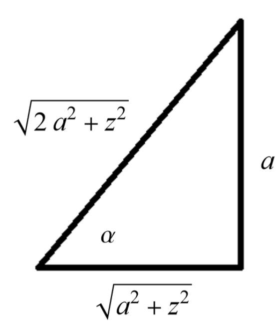
$$x = \sqrt{a^2+z^2}\tan\theta \quad \rightarrow \quad x^2+a^2+z^2 = (a^2+z^2)(\tan^2\theta+1) = (a^2+z^2)\sec^2\theta$$

$$dx = \sqrt{a^2+z^2}\sec^2\theta\, d\theta$$

As a result,

$$\begin{aligned}
\Phi_E &= \frac{qa}{4\pi\epsilon_0}\int_0^a \left[\int_{\tan^{-1}\left(\frac{0}{\sqrt{a^2+z^2}}\right)}^{\tan^{-1}\left(\frac{a}{\sqrt{a^2+z^2}}\right)} \frac{\sqrt{a^2+z^2}\sec^2\theta\, d\theta}{[(a^2+z^2)\sec^2\theta]^{3/2}}\right] dz \\
&= \frac{qa}{4\pi\epsilon_0}\int_0^a \left[\int_0^{\tan^{-1}\left(\frac{a}{\sqrt{a^2+z^2}}\right)} \frac{\sqrt{a^2+z^2}\sec^2\theta\, d\theta}{(a^2+z^2)^{3/2}\sec^3\theta}\right] dz \\
&= \frac{qa}{4\pi\epsilon_0}\int_0^a \frac{1}{a^2+z^2}\left[\int_0^{\tan^{-1}\left(\frac{a}{\sqrt{a^2+z^2}}\right)} \cos\theta\, d\theta\right] dz \\
&= \frac{qa}{4\pi\epsilon_0}\int_0^a \frac{1}{a^2+z^2}\left[(\sin\theta)\Big|_0^{\tan^{-1}\left(\frac{a}{\sqrt{a^2+z^2}}\right)}\right] dz \\
&= \frac{qa}{4\pi\epsilon_0}\int_0^a \frac{1}{a^2+z^2}\sin\tan^{-1}\left(\frac{a}{\sqrt{a^2+z^2}}\right) dz.
\end{aligned}$$

Draw the triangle implied by $\alpha = \tan^{-1}\left(a/\sqrt{a^2+z^2}\right)$ and use it to determine $\sin\alpha$.



$$\sin\alpha = \frac{a}{\sqrt{2a^2+z^2}}$$

Consequently,

$$\Phi_E = \frac{qa}{4\pi\epsilon_0}\int_0^a \frac{1}{a^2+z^2}\frac{a}{\sqrt{2a^2+z^2}}\,dz.$$

Make another substitution.

$$z = \sqrt{2}a\tan\xi \quad \to \quad 2a^2+z^2 = 2a^2(1+\tan^2\xi) = 2a^2\sec^2\xi \quad \to \quad a^2+z^2 = 2a^2\sec^2\xi - a^2 = a^2(2\sec^2\xi - 1)$$

$$dz = \sqrt{2}a\sec^2\xi\,d\xi$$

So then

$$\begin{aligned}
\Phi_E &= \frac{qa}{4\pi\epsilon_0}\int_{\tan^{-1}\left(\frac{0}{\sqrt{2}a}\right)}^{\tan^{-1}\left(\frac{a}{\sqrt{2}a}\right)} \frac{1}{a^2(2\sec^2\xi-1)}\frac{a}{\sqrt{2a^2\sec^2\xi}}(\sqrt{2}a\sec^2\xi\,d\xi) \\
&= \frac{qa}{4\pi\epsilon_0}\int_0^{\tan^{-1}\left(\frac{1}{\sqrt{2}}\right)} \frac{1}{a^2(2\sec^2\xi-1)}\frac{a}{\sqrt{2}a\sec\xi}(\sqrt{2}a\sec^2\xi\,d\xi) \\
&= \frac{q}{4\pi\epsilon_0}\int_0^{\tan^{-1}\left(\frac{1}{\sqrt{2}}\right)} \frac{\sec\xi}{2\sec^2\xi-1}\,d\xi \\
&= \frac{q}{4\pi\epsilon_0}\int_0^{\tan^{-1}\left(\frac{1}{\sqrt{2}}\right)} \frac{\sec\xi}{2\sec^2\xi-(\sec^2\xi-\tan^2\xi)}\,d\xi \\
&= \frac{q}{4\pi\epsilon_0}\int_0^{\tan^{-1}\left(\frac{1}{\sqrt{2}}\right)} \frac{\sec\xi}{\sec^2\xi+\tan^2\xi}\,d\xi \\
&= \frac{q}{4\pi\epsilon_0}\int_0^{\tan^{-1}\left(\frac{1}{\sqrt{2}}\right)} \frac{\frac{1}{\cos\xi}}{\frac{1}{\cos^2\xi}+\frac{\sin^2\xi}{\cos^2\xi}}\cdot\frac{\cos^2\xi}{\cos^2\xi}\,d\xi \\
&= \frac{q}{4\pi\epsilon_0}\int_0^{\tan^{-1}\left(\frac{1}{\sqrt{2}}\right)} \frac{\cos\xi}{1+\sin^2\xi}\,d\xi.
\end{aligned}$$

Make another substitution.

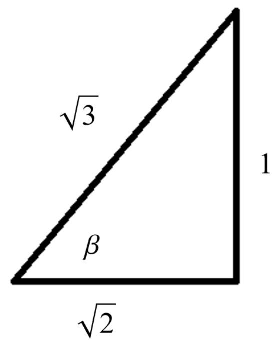
$$u = \sin\xi$$

$$du = \cos\xi\,d\xi$$

As a result,

$$\begin{aligned}
\Phi_E &= \frac{q}{4\pi\epsilon_0}\int_{\sin 0}^{\sin\tan^{-1}\left(\frac{1}{\sqrt{2}}\right)} \frac{du}{1+u^2} \\
&= \frac{q}{4\pi\epsilon_0}(\tan^{-1}u)\Big|_0^{\sin\tan^{-1}\left(\frac{1}{\sqrt{2}}\right)} \\
&= \frac{q}{4\pi\epsilon_0}\left\{\tan^{-1}\left[\sin\tan^{-1}\left(\frac{1}{\sqrt{2}}\right)\right]-\tan^{-1}0\right\}.
\end{aligned}$$

Draw the triangle implied by $\beta = \tan^{-1}(1/\sqrt{2})$ and use it to determine $\sin\beta$.



$$\sin\beta = \frac{1}{\sqrt{3}}$$

Therefore, the electric flux through the shaded side is

$$\begin{aligned}\Phi_E &= \frac{q}{4\pi\epsilon_0}\left[\tan^{-1}(\sin\beta) - \tan^{-1}0\right] \\ &= \frac{q}{4\pi\epsilon_0}\left[\tan^{-1}\left(\frac{1}{\sqrt{3}}\right) - 0\right] \\ &= \frac{q}{4\pi\epsilon_0}\left(\frac{\pi}{6}\right) \\ &= \frac{q}{24\epsilon_0}.\end{aligned}$$